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Some sharp Hölder estimates for two-dimensional elliptic equations

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Abstract We present some recent sharp estimates for the Hölder exponent of solutions of linear second order elliptic equations in divergence form with measurable coefficients. We apply such results to planar Beltrami equations, and we exhibit a mapping of the "angular stretching" type for which our estimates are attained.

Keywords Linear elliptic equation · Measurable coefficients · Hölder regularity

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1 Motivation and main results

Let Ω be a bounded open subset of $\mathbf{R}^2$ and let $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ be a weak solution of the equation

$$\operatorname{div}(A(x)\nabla u) = 0 \qquad \text{in } \Omega, \tag{1}$$

where A is a 2×2 bounded matrix-valued function, such that $A = A^T$ and such that the ellipticity condition

$$\lambda|\xi| \leq \langle \xi, A(x)\xi \rangle \leq \Lambda|\xi|^2 \tag{2}$$

holds for all $x \in \Omega$, for all $\xi \in \mathbf{R}^2$ and for some constants $0 < \lambda \leq \Lambda$. By classical works of Morrey [4], De Giorgi [2], Nash[5], u is α–Hölder continuous for some $\alpha \in (0, 1)$. See also Campanato [1] for a proof in a very general setting. In [6], Piccinini and Spagnolo obtained the following sharp estimate for α:

$$\alpha \geq L^{-1/2}, \tag{3}$$

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where $L = \Lambda/\lambda$ denotes the ellipticity constant. They also showed that if A is *isotropic*, namely if $A(x) = a(x)I$ for some measurable function satisfying $1 \le a \le L$, then

$$\alpha \ge \frac{4}{\pi} \arctan L^{-1/2}. \tag{4}$$

By an example, they showed that estimate (4) is sharp. We were thus motivated to consider the following question:

Are there other classes of matrix-valued functions A for which the classical estimate (3) *can be improved?*

In the framework of quasiconformal mappings, an answer is given from the class of matrices A satifying the condition

$$\det A \equiv 1 \qquad \text{in } \Omega. \tag{5}$$

In [8] we obtained the following result:

Theorem 1 ([8]) *Let* $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *be a weak solution of* (1), *and let* A *satisfy* (2) *and* (5). *Then*

$$\alpha \ge \left(\sup_{S_\rho(x) \subset \Omega} \frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \langle n, An \rangle \right)^{-1}, \tag{6}$$

where $S_\rho(x)$ *is the circle centered at* x *having radius* ρ *and* n *is the outer unit normal to* $S_\rho(x)$. *This estimate is sharp.*

Our next natural question was: *Can one unify estimates* (3)–(4)–(6) *in a single formula*?
In this direction, we obtained the following result.

Theorem 2 ([10]) *Let* $u \in W^{1,2}_{\mathrm{loc}}(\Omega)$ *be a weak solution of* (1) *and let* A *satisfy* (2). *Then,* u *is* α*-Hölder continuous with* $\alpha \ge \gamma(A)$, *where*

$$\gamma(A) = \left(\sup_{S_\rho(x) \subset \Omega} \inf_{\varphi, \psi \in \mathcal{B}_{x,\rho}} \sqrt{\frac{\sup \varphi}{\inf \psi}} \frac{\frac{1}{|S_\rho(x)|} \int_{S_\rho(x)} \sqrt{\frac{\psi}{\varphi}} \frac{\langle n, A\, n\rangle}{\sqrt{\det A}}}{\frac{4}{\pi} \arctan \left(\frac{\inf_{S_\rho(x)} \det A/\varphi\psi}{\sup_{S_\rho(x)} \det A/\varphi\psi} \right)^{1/4}} \right)^{-1}. \tag{7}$$

Here, $S_\rho(x)$ *is the circle centered at* x *having radius* ρ, n *is the outer unit normal to* $S_\rho(x)$ *and* $\mathcal{B}_{x,\rho}$ *is the set of measurable functions on* $S_\rho(x)$ *which are bounded from above and from below away from zero.*

A key ingredient in the proof of Theorem 2 is a family of sharp Wirtinger inequalities obtained in [9], which generalize the Wirtinger inequality in [6]. Despite of its complicated form, the estimate of Theorem 2 is sharp, in the sense that it is attained for particular choices of u and A. Indeed, we have:

Theorem 3 ([10]) *For every* $\tau \in [0, 1]$ *there exist bounded matrix-valued functions* $A_\tau : \mathbf{R}^2 \setminus \{0\} \to \mathcal{M}(2 \times 2)$ *satisfying (2) and corresponding functions* $u_\tau \in W^{1,2}_{\mathrm{loc}}(\mathbf{R}^2)$ *such that:*

(i) A_0 *is isotropic,* A_1 *has unit determinant and for every* $\tau \in (0, 1)$ *the matrix field* A_τ *is neither isotropic nor has unit determinant;*
(ii) u_τ *satisfies Eq.* (1) *with* $A = A_\tau$*;*
(iii) u_τ *is* α*-Hölder continuous with* $\alpha = \gamma(A_\tau)$.

We briefly sketch the construction of A_τ and u_τ. For every $\theta \in \mathbf{R}$ let

$$J(\theta) = \begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}.$$

For every $M > 1$, let

$$c = c(M,\tau) = \frac{2}{1+M^{-\tau}}, \qquad d = d(M,\tau) = \frac{4}{\pi}\arctan M^{-(1-\tau)/2}.$$

We define the intervals

$$I_1 = [0, \frac{c\pi}{2}), \qquad I_2 = [\frac{c\pi}{2}, \pi), \qquad I_3 = [\pi, \pi + \frac{c\pi}{2}), \qquad I_4 = [\pi + \frac{c\pi}{2}, 2\pi).$$

For every $\tau \in [0,1]$ we let A_τ be the symmetric matrix-valued function defined for every $z \in \mathbf{R}^2 \setminus \{0\}$ by

$$\begin{aligned} A_\tau(z) &= (k_{\tau,1}(\arg z) - k_{\tau,2}(\arg z))\frac{z \otimes z}{|z|^2} + k_{\tau,2}(\arg z)\mathbf{I} \\ &= \begin{pmatrix} k_{\tau,1}\cos^2\theta + k_{\tau,2}\sin^2\theta & (k_{\tau,1} - k_{\tau,2})\sin\theta\cos\theta \\ (k_{\tau,1} - k_{\tau,2})\sin\theta\cos\theta & k_{\tau,1}\sin^2\theta + k_{\tau,2}\cos^2\theta \end{pmatrix} \\ &= JK_\tau J^*, \end{aligned}$$

where $k_{\tau,1}, k_{\tau,2}$ are the piecewise constant, 2π-periodic functions defined by

$$k_{\tau,1}(\theta) = \begin{cases} 1, & \text{if } \theta \in I_1 \cup I_3, \\ M, & \text{if } \theta \in I_2 \cup I_4, \end{cases}$$

$$k_{\tau,2}(\theta) = \begin{cases} 1, & \text{if } \theta \in I_1 \cup I_3, \\ M^{1-2\tau}, & \text{if } \theta \in I_2 \cup I_4 \end{cases}$$

and $K_\tau = \operatorname{diag}(k_{\tau,1}, k_{\tau,2})$. Let $\Theta_\tau : \mathbf{R} \to \mathbf{R}$ be the 2π-periodic Lipschitz function defined in $[0, 2\pi)$ by

$$\Theta_{\tau,1}(\theta) = \begin{cases} \sin[d(c^{-1}\theta - \pi/4)], & \theta \in I_1, \\ M^{-(1-\tau)/2}\cos[d(c^{-1}M^\tau(\theta - c\pi/2) - \pi/4)], & \theta \in I_2, \\ -\sin[d(c^{-1}(\theta - \pi) - \pi/4)], & \theta \in I_3, \\ -M^{-(1-\tau)/2}\cos[d(c^{-1}M^\tau(\theta - \pi - c\pi/2) - \pi/4)] & \theta \in I_4. \end{cases}$$

In [10] we prove Theorem 3 by showing that there exists $M_0 > 1$ such that

$$\gamma(A_\tau) = \frac{d}{c} \tag{8}$$

for every $M \in (1, M_0^{1/\tau})$ if $\tau > 0$ and with no restriction on M if $\tau = 0$. Furthermore, the function $u_\tau = |z|^{d/c}\,\Theta_1(\arg z)$ is a weak solution of (1) with $A = A_\tau$.

2 Application to Beltrami equations

As already mentioned, our results are of interest in the context of Beltrami equations and quasiconformal mappings, see, e.g., [3]. Indeed, the following correspondence is well-known.

Lemma 1 *Let $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{C})$ satisfy the Beltrami equation*

$$\overline{\partial} f = \mu \partial f + \nu \overline{\partial f} \qquad \text{in } \Omega, \tag{9}$$

where $\mu, \nu \in L^\infty(\Omega, \mathbf{C})$ satisfy $|\mu| + |\nu| \le \kappa < 1$ a.e. in Ω. Let $B_{\mu,\nu}$ be the bounded matrix-valued function defined in terms of the Beltrami coefficients μ, ν by

$$B_{\mu,\nu} = \frac{1}{\Delta_1}\left(\begin{bmatrix} |1-\mu|^2 & -2\Im(\mu-\nu) \\ -2\Im(\mu+\nu) & |1+\mu|^2 \end{bmatrix} - |\nu|^2 \mathbf{I}\right),$$

where $\Delta_1 = |1+\nu|^2 - |\mu|^2$ and let $\widehat{B}_{\mu,\nu}$ be defined by

$$\widehat{B}_{\mu,\nu} = \frac{1}{\Delta_2}\left(\begin{bmatrix} |1-\mu|^2 & -2\Im(\mu+\nu) \\ -2\Im(\mu-\nu) & |1+\mu|^2 \end{bmatrix} - |\nu|^2 \mathbf{I}\right)$$

with $\Delta_2 = |1-\nu|^2 - |\mu|^2$. Then $\Re(f)$ is a weak solution of (1) with $A = B_{\mu,\nu}$ and $\Im(f)$ is a weak solution of (1) with $A = \widehat{B}_{\mu,\nu}$.

In particular, solutions of (9) with $\Im(\nu) = 0$ correspond to solutions of (1) with a symmetric coefficient matrix. Furthermore, when $\nu = 0$ we have the unit determinant case, and when $\mu = 0$ we have the isotropic case.

Theorem 4 *Let $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{C})$ satisfy the Beltrami Eq. (9) with $\Im(\nu) = 0$. Then, f is α-Hölder continuous with $\alpha \ge \beta(\mu, \nu)$, where $\beta(\mu, \nu)$ is defined by*

$$\begin{aligned}\beta(\mu,\nu)^{-1} = & \sup_{S_\rho(x)\subset\Omega} \inf_{\varphi,\psi\in\mathcal{B}_{x,\rho}} \sqrt{\frac{\sup\varphi}{\inf\psi}} \\ & \left\{ \frac{1}{|S_\rho(x)|}\int_{S_\rho(x)} \sqrt{\frac{\psi}{\varphi}} \frac{|1-\overline{n}^2\mu|^2 - \nu^2}{\sqrt{1-(|\mu|+\nu)^2}\sqrt{1-(|\mu|-\nu)^2}} \mathrm{d}\sigma \right. \\ & \left. \times \left(\frac{4}{\pi}\arctan\left(\frac{\inf_{S_\rho(x)} \frac{(1-\nu)^2-|\mu|^2}{(1+\nu)^2-|\mu|^2}/\varphi\psi}{\sup_{S_\rho(x)} \frac{(1-\nu)^2-|\mu|^2}{(1+\nu)^2-|\mu|^2}/\varphi\psi}\right)^{1/4}\right)^{-1} \right\},\end{aligned}$$

where $\mathcal{B}_{x,\rho}$ denotes the set of positive functions in $L^\infty(S_\rho(x))$ which are bounded from above and from below away from zero, and where n denotes complex number corresponding to the outer unit normal of $S_\rho(x)$. This estimate is sharp, in the sense that it is attained for suitable choices of μ and f.

In particular, when $\nu = 0$, Theorem 4 implies the following result, which it is interesting to compare with an estimate of Reich and Walczak [7] for the conformal modulus of rings:

Corollary 1 *Let $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbf{C})$ satisfy the Beltrami Eq. (9) with $\nu = 0$. Then, f is α-Hölder continuous with $\alpha \ge \beta(\mu)$, where $\beta(\mu)$ is defined by*

$$\beta(\mu) = \left(\sup_{S_\rho(x)\subset\Omega} \frac{1}{|S_\rho(x)|}\int_{S_\rho(x)} \frac{|1-\overline{n}^2\mu|^2}{1-|\mu|^2}\right)^{-1}$$

and where n denotes complex number corresponding to outer unit normal to $S_\rho(x)$.

The sharpness of Theorem 4 may be seen as follows. For every $\tau \in [0, 1]$ we let $\mu_{0,\tau}, \nu_{0,\tau}$: $\mathbf{R} \to \mathbf{R}$ be the bounded, piecewise constant, 2π-periodic functions defined in $[0, 2\pi)$ by

$$\mu_{0,\tau}(\theta) = \begin{cases} 0, & \text{if } \theta \in I_1 \cup I_3, \\ (M - M^{1-2\tau})/(1 + M + M^{1-2\tau} + M^{2(1-\tau)}), & \text{if } \theta \in I_2 \cup I_4 \end{cases}$$

and

$$\nu_{0,\tau}(\theta) = \begin{cases} 0, & \text{if } \theta \in I_1 \cup I_3, \\ (M^{2(1-\tau)} - 1)/(1 + M + M^{1-2\tau} + M^{2(1-\tau)}), & \text{if } \theta \in I_2 \cup I_4 \end{cases}$$

and we set

$$\mu_\tau(z) = -\mu_{0,\tau}(\arg z)\, z\bar{z}^{-1}, \qquad \nu_\tau(z) = -\nu_{0,\tau}(\arg z).$$

Then, the following holds.

Proposition 1 *Let B the unit disk in $\mathbf{R}^2$ and let $f_\tau \in W^{1,2}(B, \mathbf{C})$ be defined in $B \setminus \{0\}$ by*

$$f_\tau(z) = |z|^{d/c}\left(\Theta_{\tau,1}(\arg z) + i\Theta_{\tau,2}(\arg z)\right).$$

Then f_τ satisfies (9) with $\mu = \mu_{0,\tau}$ and $\nu = \nu_{0,\tau}$. Furthermore, there exists $M_0 > 1$ such that

$$\beta(\mu_\tau, \nu_\tau) = \frac{d}{c}$$

for every $M \in (1, M_0^{1/\tau})$ if $\tau > 0$ and with no restriction on M if $\tau = 0$.

Here c, d are the constants defined in the previous section. We note that the mapping f_τ defined above is of the "angular stretching" type.

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